Stochastic integration, Ito's formula

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Now, we are finally ready to introduce Itô's Stochastic Integration.

Def Let $M \in H^2$. $\mathcal{L}^2(M) := \{ K$ - progressively measurable, such that $\|K\|_M^2 := E\left(\int_0^\infty K_s^2 \, d\langle M,M\rangle_s\right) < \infty \}$

Remark. It is L^2 with respect to measure on $\mathbb{R} \times \Omega$ given by $P_M(A) := E\left(\int_0^\infty 1_A(s,\omega) \, d\langle M, M\rangle_s(\omega)\right)$.

Thm. Let $M \in H^2$, $K \in L^2(M)$

Then $\exists$ unique $K \cdot M \in H_0^2$:

$\langle K\cdot M, N\rangle = K \cdot \langle M, N\rangle = \int K \, d\langle M, N\rangle \quad \forall N \in H^2$.

The map $K \to K \cdot M$ is an isometry from $\mathcal{L}^2(M)$ into H_0^2.

Notation - $(K \cdot M)_t = \int_0^t K_s \, dM_s$ - stochastic (Itô) integral.

$$\left\langle \int_0^t K_s \, dM_s, N_t \right\rangle_t = \int_0^t K_s \, d\langle M, N\rangle_s.$$

Proof Uniqueness: $L, L' \in H_0^2$, $\langle L, N\rangle = \langle L', N\rangle = \int K \, d\langle M, N\rangle$ $\forall N \in H^2$.

Then for $N = L - L'$, $\langle L, L-L'\rangle_t = \langle L', L-L'\rangle_t \ \forall t$,

i.e. $\langle L-L', L-L'\rangle_\infty = 0 \Rightarrow L - L' \equiv 0$ ∎

$$\Downarrow$$

$$E\left((L-L')_\infty^2\right) = E\left(\underset{0 \in H_0^2}{\underbrace{(L-L')_0^2}}\right) + E\left(\underset{0}{\underbrace{\langle L-L', L-L'\rangle_\infty}}\right) = 0$$

Existence. First, assume $M \in H_0^2$.

Then $\forall N \in H_0^2$, by Kunita-Watanabe inequality:

$$E(K\cdot\langle M,N\rangle_\infty) = E\left(\int_0^\infty K\, d\langle M,N\rangle_t\right) \le E\left(\int_0^\infty d\langle N,N\rangle_t\right)^{1/2} \cdot \quad (=\langle N,N\rangle_\infty = E(N_\infty^2))$$

$$E\left(\int_0^\infty K^2 d\langle M,M\rangle_t\right)^{1/2} =$$

$$= \|N\|_{H^2} \|K\|_{L^2(M)}$$

So $N \to E(K\cdot\langle M,N\rangle_\infty)$ is continuous on H_0^2.

So $\exists L \in H_0^2: \ \forall N \in H_0^2 \ \underbrace{E(L_\infty N_\infty)}_{\langle L,N\rangle_{H^2}} = E(K\cdot\langle M,N\rangle_\infty)$.

If T - stopping time, then

$$E(L_T N_T) = E(E(L_\infty | \mathcal{F}_T) N_T) =$$

$$E(L_\infty N_T) = E(L_\infty N_\infty^T) \overset{\text{property}}{=} E(K\cdot\langle M,N^T\rangle_\infty) =$$

$(N_t^T \in H_0^2)$

$$E((K\cdot\langle M,N\rangle^T)_\infty) = E(K\cdot\langle M,N\rangle_T) = E\left(\int_0^T K_s\, d\langle M,N\rangle_s\right)$$

So $L_t N_t - K_t\langle M,N\rangle_t$ - martingale.

So $\langle L,N\rangle_t = K\cdot\langle M,N\rangle_t$. We can take

$K\cdot M := L$.

General $N \in H^2$: $N = \tilde{N} + N_0$, where $\tilde{N} := N - N_0 \in H_0^2$.

Then $\langle M,N\rangle_t = \langle M,\tilde{N}\rangle_t$, so the identity
$\langle L,N\rangle_t = \langle L,\tilde{N}\rangle_t$

$\langle L,N\rangle_t = K\cdot\langle M,N\rangle_t$ holds.

Finally, if $M \in H^2$, take $\tilde{M} := M - M_0 \in H_0^2$,

$\langle M,N\rangle_t = \langle \tilde{M},N\rangle_t \ \forall t$.

so can take $K\cdot M := K\cdot\tilde{M}$.

Isometry: $M \in H_0^2$,

$$\|K\cdot M\|_{H^2}^2 = E((K\cdot M)_\infty^2) = E(\langle K\cdot M, K\cdot M\rangle_\infty) =$$

$$E\left(\int_0^\infty K_t\, d\langle M, K\cdot M\rangle_t\right) = E\left(\int_0^\infty K_t^2\, d\langle M,M\rangle_t\right) =$$

$$\|K\|_{L^2(M)}^2$$

Def $(K\cdot M)_t =: \int_0^t K_s \, dM_s$ – Itô stochastic integral.

Properties ① Let $K = \sum K_i 1_{(t_i, t_{i+1}]}$ - step function.

Then $(K\cdot M)_t = \sum_{i=0}^{n-1} K_i (M_{t_{i+1}} - M_{t_i}) + K_n (M_t - M_{t_n})$ for $t_n \le t < t_{n+1}$.

For proof, notice that for $K = 1_{(a,b]}$,

$(K\cdot M)_t = (M_t^b - M_t^a)$

$\langle K\cdot M, N\rangle_\infty = \langle M, N\rangle_b - \langle M, N\rangle_a$

$K\cdot \langle M, N\rangle = \int_0^\infty 1_{(a,b]} \, d\langle M, N\rangle = \langle M, N\rangle_b - \langle M, N\rangle_a$.

② Associativity:

$K \in \mathcal{L}^2(M)$, $H \in \mathcal{L}^2(K\cdot M)$. Then

$HK \in \mathcal{L}^2(M)$, and $(HK)\cdot M = H\cdot(K\cdot M)$

(i.e. $\int HK \, dM = \int H \, d\, K\cdot M$).

Proof $\langle K\cdot M, K\cdot M\rangle = K^2 \cdot \langle M, M\rangle$ so

$\int H^2 K^2 \, d\langle M, M\rangle = \int H^2 \, d\langle K\cdot M, K\cdot M\rangle < \infty$ and $HK \in \mathcal{L}^2(M)$.

By associativity of the Stilties integral, for $N \in H^2$, we have

$\langle (HK)\cdot M, N\rangle = HK\cdot\langle M, N\rangle = H\cdot(K\cdot\langle M, N\rangle) =$
$H\cdot\langle K\cdot M, N\rangle = \langle H\cdot(K\cdot M), N\rangle$, so it holds by uniqueness ∎

③ If T is a stopping time, $K \in \mathcal{L}^2(M)$, then

$K\cdot M^T = K 1_{[0,T]} \cdot M = (K\cdot M)^T$.

Proof Observe: $M^T = 1_{[0,T]}\cdot M$

(since for $N \in H^2$, $\langle M^T, N\rangle = \langle M, N\rangle^T = 1_{[0,T]} \cdot \langle M, N\rangle$).

So, by associativity,

$K\cdot M^T = (K \; 1_{[0,T]})\cdot M$

and $(K\cdot M)^T = (1_{[0,T]} \; K)\cdot M$ ∎

The previous property can be used to integrate wrt local martingales/unbounded quadratic variation (like Brownian Motion).

Def M - continuous local martingale.

$\mathcal{L}^2_{loc}(M) := \{ k$-progressively measurable; $\exists T_n \uparrow \infty$ - stopping times: $E\left(\int_0^{T_n} k_s^2 \, d\langle M, M\rangle_s\right) < \infty$.

Remark. An equivalent definition:

$\mathcal{L}^2_{loc}(M) = \{ k$-progressively measurable, $\forall t \int_0^t k_s^2 \, d\langle M, M\rangle_s < \infty \}$

Thm. $\forall k \in \mathcal{L}^2_{loc}(M)$ $\exists$ unique local martingale $k \cdot M$, such that $(k \cdot M)_0 \equiv 0$, and for any continuous local martingale N,

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle.$$

Remark. Unlike the case $M \in H^2$, even if M is a martingale, $(H \cdot M)_t = \int_0^t H_s \, dM_s$ does not have to be!

Proof. As before, assume $M_0 \equiv 0$ (can consider $\tilde{M} = M - M_0$ for general M, $k \cdot M := k \cdot \tilde{M}$, since $\forall N$ $\langle M, N\rangle = \langle \tilde{M}, N\rangle$).

$T_n := \inf\{ t \geq 0 : \int_0^t (1 + H_s^2) \, d\langle M, M\rangle_s \geq n\}$ - stopping times, $T_n \uparrow \infty$.

$\langle M^{T_n}, M^{T_n}\rangle_t \leq n$, so $M^{T_n} \in H^2$.

Also $\int_0^\infty H_s^2 \, d\langle M^{T_n}, M^{T_n}\rangle = \int_0^{T_n} H_s^2 \, d\langle M, M\rangle_s \leq n \Rightarrow H \in \mathcal{L}^2(M^{T_n})$

Also, by stopping time property of integral:

$$H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n} \text{ if } m > n.$$

So $\exists (H \cdot M)_t := (H \cdot M^{T_n})_t$ if $t < T_n$.

It is continuous, adapted ($(H \cdot M)_t = \lim_{n \to \infty} (H \cdot M^{T_n})_t$).

$(H \cdot M)_t^{T_n} = (H \cdot M^{T_n})_t$, so $(H \cdot M)_t$ - continuous local martingale.

Now take any continuous local martingale N,

S_n - stopping times : N^{S_n} - bounded, $S_n \uparrow \infty$.

$T_n' := \min(S_n, T_n) \uparrow \infty$.

$N^{T_n'} \in H^2$, so $\langle H\cdot M, N\rangle^{T_n'} = (\langle (H\cdot M)^{T_n'}, N^{T_n'}\rangle =$

$$= \langle H\cdot M^{T_n'}, N^{T_n'}\rangle = H\cdot \langle M^{T_n'}, N^{T_n'}\rangle = H\cdot\langle M, N\rangle^{T_n'}$$

Since $T_n' \uparrow \infty$, we get $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. $\blacksquare$

Def. A progressively measurable process K is locally bounded if $\exists\, T_n \uparrow \infty$ - stopping times, $C_n > 0$:

$$|K^{T_n}| \le C_n.$$

Remarks ① Any continuous K is locally bounded, take

$$T_n = \inf\{t : |K_t| \ge n\}.$$

② K - locally bounded $\Rightarrow$ $K \in \mathcal{L}^2_{loc}(M)$ $\forall M$ - continuous local martingale.

Def K - locally bounded, $X = \underbrace{M}_{\text{local martingale}} + \underbrace{A}_{\text{process of bounded variation}}$ - semimartingale.

$$K\cdot X = \int K_s \, dX_s = \underbrace{\int K_s \, dM_s}_{\text{Stochastic}} + \underbrace{\int K_s \, dA_s}_{\text{Stilties.}}$$

Thm (Dominated convergence).

X - continuous semimartingale, $K^{(n)}$ - sequence of locally bounded processes, K - locally bounded process, $\left(\begin{array}{l} |K^{(n)}| \le K. \\ K^{(n)} \xrightarrow{a.s.} 0 \end{array}\right)$ Then $K^{(n)}\cdot X \to 0$ in probability

Proof. $X = M + A$. For dA - usual dominated convergence

To verify convergence dM, fix t and consider

$$T_m := \min\left(t, \inf\left\{r \in [0,t] : \int_0^r K_s^2 \, d\langle M, M\rangle_s \ge m\right\}\right)$$

Then $E\left(\left(\int^{T_m} K_s^{(n)} dM_s\right)^2\right) = E\left(\int^{T_m} (K^{(n)})^2 \, d\langle M, M\rangle\right)$

Then $E\left(\left(\int_0^{T_m} K_s^{(n)} dM_s\right)^2\right) = E\left(\int_0^{T_m} (K_s^{(n)})^2 d\langle M, M\rangle_s\right)$

By usual dominated convergence,

$$\int_0^{T_m} (K_s^{(n)})^2 d\langle M, M\rangle_s \to 0 \quad a.s.$$

Another application of dominated by $\int_0^{T_m} K_s^2 d\langle M, M\rangle_s$ convergence, gives that

$$E\left(\left(\int_0^{T_m} K_s^{(n)} dM_s\right)^2\right) \xrightarrow[n\to\infty]{} 0 .$$

Observe now that $P(T_m = t) \xrightarrow[m\to\infty]{} 1$, so

$$E\left(\left(\int_0^t K_s^{(n)} dM_s\right)^2\right) \to 0 \Rightarrow P\text{-}\lim \int_0^t K_s^{(n)} dM_s = 0$$

Corollary. Let X be a continuous semimartingale, K - adapted continuous process. Then $\forall t > 0$, $\Delta_n = \{0 = t_0^n < t_1^n < \dots < t_{m_n}^n = t\}$ - subdivisions, $|\Delta_n| \to 0$, we have

$$\int_0^t K_s dX_s = \lim_{n\to\infty} \sum_{j=0}^{m_n - 1} K_{t_j^n} (X_{t_{j+1}^n} - X_{t_j^n}) .$$

Proof Consider step functions

$$K^{(n)} := \sum K_{t_i^n} 1_{(t_i^n, t_{i+1}^n]} .$$

Then $K^{(n)} - K \to 0$ pointwise, bounded by $H_t := 2\max_{s\le t} |K_s|$.

And $\int K_s^{(n)} dX_s = \sum_{j=0}^{m_n - 1} K_{t_j^n} (X_{t_{j+1}^n} - X_{t_j^n})$.

Proposition (pre-Itô formula, integration by parts)

Let X, Y - be continuous semimartingales.

Then $X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t$

Let X, Y - be continuous semimartingales.

Then $$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \underline{\langle X, Y \rangle_t}$$

In particular, $X_t^2 = X_0^2 + 2\int_0^t X_s \, dX_s + \langle X, X\rangle_t$.

Example. $\displaystyle\int_0^t B_s \, dB_s = B_t^2 - t$

Proof. By polarization, enough to prove for $X = Y$.

For a subdivision Δ, we have

$$\sum (X_{t_{i+1}} - X_{t_i})^2 = X_t^2 - X_0^2 - 2\sum X_{t_i}(X_{t_{i+1}} - X_{t_i})$$

Now let $|\Delta| \to 0$. ∎

Theorem (Itô's formula)

Let $X = (X^1, \dots, X^n)$ - continuous semimartingales, $F \in C^2(\mathbb{R}^n \to \mathbb{R})$.

Then

$$F(X_t) = F(X_0) + \sum_{j=1}^n \int_0^t \frac{\partial F}{\partial X^j}(X_s) \, dX_s^j + \frac{1}{2}\sum_{i,j} \int_0^t \frac{\partial^2 F}{\partial X^i \partial X^j}(X_s) \, d\langle X^i, X^j\rangle_s$$

Proof. By using stopping time, can assume that each X^j is bounded. Then polynomials are dense in $C^2(\overline{B(0,R)})$, $\operatorname{Ran}(X) \subset B(0,R)$, so enough to prove for them.

Induction on degree: Base: $\deg F = 0$ - obvious.

Step. If true for F, true for $x^k F$ by integration by parts:

$$X_t^k F(X_t) = X_0^k F(X_0) + \int_0^t X_s^k \, dF(X_s) + \int_0^t F(X_s) \, dX_s^j + \langle X^k, F(X)\rangle_t$$

$$= X_0^k F(X_0) + \sum_{j \neq k} \int_0^t X_s^k \frac{\partial F}{\partial X^j}(X_s) \, dX_s^j + \int_0^t \left(X^k \frac{\partial F}{\partial X^k} + F\right) dX_s^k + \sum_{i,j} \int_0^t \frac{\partial^2 X^k F}{\partial X_i \partial X_j}(X_s) \, d\langle X^i, X^j\rangle_s =$$

Differential notation:

$$dF(X_t) = \sum \frac{\partial F}{\partial X_i}(X_t) \, dX_t^j + \frac{1}{2}\sum \frac{\partial^2 F}{\partial X_i \partial X_j}(X_t) \, d\langle X^i, X^j\rangle_t$$

Corollary. $F: \mathbb{R}^2 \to \mathbb{C}$, $\in C^2$.

$\frac{\partial F}{\partial y} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$ Let M be a local martingale

Then so is $F(M, \langle M, M\rangle)$

Proof. Apply Itô, the only non martingale term: $\int_0^t \frac{\partial F}{\partial y} d\langle M, M\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2} d\langle M, M\rangle_s = 0$

Examples. $f(x,y) = x^2 - y$ $(M_t^2 - \langle M, M\rangle_t)$

$f(x,y) = \underbrace{e^{\lambda x - \frac{\lambda^2}{2}y}}$ $\left(e^{\lambda M_t - \frac{\lambda^2}{2}\langle M, M\rangle_t}\right.$ - exponential transform).

Corollary B - d-dim BM ($B = (B_t^1, \ldots, B_t^d)$ - d independent copies of BM)

$f \in C^2(\mathbb{R}_+ \times \mathbb{R}^d)$, then

$M_t^f := f(t, B_t) - \int_0^t \left(\frac{1}{2}\Delta f + \frac{\partial f}{\partial t}\right)(s, B_s)\, ds$ - local martingale

In particular, if $\frac{1}{2}\Delta f + \frac{\partial f}{\partial t} = 0$ (Heat equation!) then $f(t, B_t)$ - local martingale.

Or if f is harmonic in $\mathbb{R}^d$ and t-independent.

$$\langle B^i, B^j\rangle_t = \delta_{ij} t$$

Complex notation.

$Z_t = X_t + i Y_t$, $\langle Z, Z\rangle_t = \langle X, X\rangle_t - \langle Y, Y\rangle_t + 2i\langle X, Y\rangle_t$

$F \in C^2(\mathbb{C}) \Rightarrow$

$$F(Z_t) = F(z_0) + \int_0^t \frac{\partial F}{\partial z}(z_s)\, dz_s + \int_0^t \frac{\partial F}{\partial \bar{z}_s}(z_s)\, d\bar{z}_s + \frac{1}{4}\int_0^t \Delta F(z_t)\, d\langle z, \bar{z}\rangle_t$$

In particular, Z - local martingale, F - harmonic $\Rightarrow$ $F(Z_t)$ - local martingale.